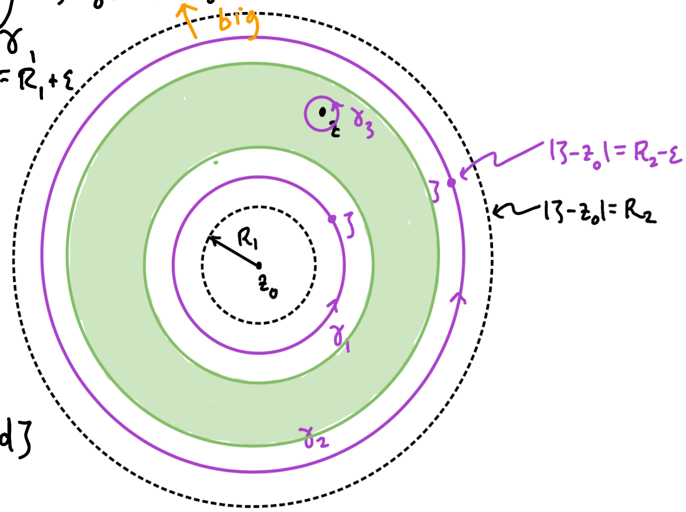
$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}d\zeta$$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\underbrace{(\zeta - z_0)}_{\text{big}} - (z - z_0)}d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{(\zeta - z_0) - \underbrace{(z - z_0)}_{\text{big}}}d\zeta$$

γ_2: $|\zeta - z_0| = R_2 - \varepsilon$ $\qquad$ γ_1: $|\zeta - z_0| = R_1 + \varepsilon$



$$= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z_0}\ \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}\, d\zeta$$

- $\left|\frac{z - z_0}{\zeta - z_0}\right| \le \frac{R_2 - 2\varepsilon}{R_2 - \varepsilon} < 1$ on γ_2

$$+ \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{(z - z_0)}\ \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}}\, d\zeta$$

- $\left|\frac{\zeta - z_0}{z - z_0}\right| \le \frac{R_1 + \varepsilon}{R_1 + 2\varepsilon} < 1.$

$$= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(\zeta - z_0)^n}\, d\zeta + \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{z - z_0}\sum_{k=0}^{\infty}\frac{(\zeta - z_0)^k}{(z - z_0)^k}\, d\zeta$$

uniform conv to interchange ints with sums
also factor powers of $(z - z_0)$

$$= \sum_{n=0}^{\infty}(z - z_0)^n \underbrace{\left(\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta\right)}_{a_n} + \sum_{k=0}^{\infty}\frac{1}{(z - z_0)^{k+1}}\ \frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta - z_0)^k\, d\zeta$$

$(m = k+1)$ $\qquad a_{-m} = \frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta - z_0)^{m-1}\, d\zeta$

On the next two pages we use Laurent series to classify *isolated singularities,* and give equivalent geometric conditions which characterize the three kinds: removable singularities, poles, essential singularities. And we will revert to the text's lettering for the coefficients of the positive and negative powers in a Laurent series. $f(z)$ has an *isolated singularity at* z_0 means that there is some radius $r > 0$ so that f is analytic in the punctured disk $D(z_0, r) \setminus \{z_0\}$ We write the Laurent series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m}$$

a_n $\qquad$ (a_{-m})

$$b_m = \frac{1}{2\pi i}\int_{\gamma} f(\zeta)(\zeta - z_0)^{m-1}\, d\zeta$$

Wed!!

Isolated singularities table

Let f analytic in $D(z_0;r)\setminus\{z_0\}$, some $r>0$

type of singularity @ z_0	definition: Laurent series definition	geometric characterization (behavior of f near z_0)
removable (because f extends to be analytic @ z_0)	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ (no negative powers in L.S.)	any of: ① $\lim_{z\to z_0} f(z)=L\in\mathbb{C}$ exists ⇓② ② $\lvert f(z)\rvert\le M$ $\forall\ 0<\lvert z-z_0\rvert\le\rho$, some $0<\rho<r$ ⇓③ ③ $\lim_{z\to z_0} f(z)(z-z_0)=0$
pole (North pole!) of order $N\in\mathbb{N}$; simple pole if $N=1$	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\sum_{m=1}^{N}\frac{b_m}{(z-z_0)^m}$ with $b_N\neq 0$	① $\lim_{z\to z_0} f(z)=\infty$ (the north pole of the Riemann sphere)

① $L=a_0$

② since f has a limit at z_0.

③ $\lim_{z\to z_0}\lvert f(z)\rvert\lvert z-z_0\rvert\le M\cdot 0$

④ Laurent coef's b_m for negative powers $m\ge1$

$$b_m=\frac{1}{2\pi i}\int_{\lvert z-z_0\rvert=\rho}\underbrace{f(\zeta)(\zeta-z_0)^{m-1}}_{f(\zeta)(\zeta-z_0)(\zeta-z_0)^{m-2}}d\zeta$$

$$\lvert b_m\rvert\le\frac{1}{2\pi}\underbrace{\max_{\lvert\zeta-z_0\rvert=\rho}\{\lvert f(\zeta)\rvert\lvert\zeta-z_0\rvert\}}_{\to 0}\ \underbrace{\rho^{m-2}\,2\pi\rho}_{\text{at least bd}\ \le 2\pi,\ m\ge1}\to 0$$

$$f(z)=\frac{1}{(z-z_0)^N}\left[b_N+b_{N-1}(z-z_0)+\dots+b_1(z-z_0)^{N-1}+a_0(z-z_0)^N+\dots\right]$$

$$f(z)=\frac{1}{(z-z_0)^N}g(z),\quad g\ \text{analytic @ } z_0$$

$\lim_{z\to z_0} f(z)=\infty$ ①

⇓③

② $\exists\ N\in\mathbb{N}$ s.t. $g(z)=(z-z_0)^N f(z)$ has a removable singularity at $z=z_0$, with $g(z_0)\neq 0$

③ Let $\lim_{z\to z_0} f(z)=\infty$

$\Rightarrow \lim_{z\to z_0}\frac{1}{f(z)}=0$

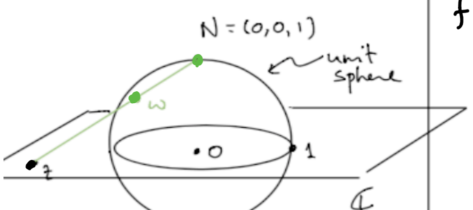
so $\frac{1}{f(z)}$ has a remov. sing @ $z=z_0$

$$\frac{1}{f(z)}=\sum_{n=N}^{\infty}a_n(z-z_0)^n\qquad a_N\neq0$$

$$\frac{1}{f(z)}=(z-z_0)^N h(z)\qquad h(z_0)\neq0,\ h\ \text{analytic}$$

$$\Rightarrow f(z)=\frac{1}{(z-z_0)^N}g(z)\qquad g(z)=\frac{1}{h(z)}$$

②!



$z=x+iy$

$$w=\left(\frac{2x}{1+x^2+y^2},\frac{2y}{1+x^2+y^2},\frac{-1+x^2+y^2}{1+x^2+y^2}\right)$$

$$x+iy=\frac{w_1}{1-w_3}+i\frac{w_2}{1-w_3}\quad(w_3\neq0)$$

$\lvert z\rvert\to\infty$ iff $w\to(0,0,1)$

type of singularity @ z_0	Laurent series def.	geometric characterization
essential singularity	$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$ with $b_{m_j} \neq 0$ for some seq. $\{m_j\} \to \infty$.	$\forall\, 0<\rho<r$, $\overline{f(D(z_0;\rho) \setminus \{z_0\})} = \mathbb{C}$! (In fact, more is true and is called "Picard's Thm": $f(D(z_0;\rho) \setminus \{z_0\})$ contains all of $\mathbb{C}$ except for at most a single point ! $\forall\, 0<\rho<r$) e.g. $f(z) = e^{\frac{1}{z}}$ @ $z_0 = 0$, $f(D(0;\rho) \setminus \{0\}) = \mathbb{C} \setminus \{0\}$ $\forall \rho > 0$. !!

※ to be cont'd

① Logic! Assume closure statement does not hold. This will imply z_0 is actually remov sing or a pole.

So $\exists\, \rho > 0$ and $D(w_0;\varepsilon)$ s.t. $f\big(D(z_0;\rho) \setminus \{z_0\}\big) \cap D(w_0;\varepsilon) = \emptyset$

Consider $g(z) = \dfrac{1}{f(z) - w_0}$ on $D(z_0;\rho) \setminus \{z_0\}$

$|g(z)| \le \dfrac{1}{\varepsilon}$ on $D(z_0;\rho) \setminus \{z_0\}$

$\Rightarrow$ g has remov sing @ z_0

$$\frac{1}{f(z) - w_0} = (z-z_0)^N h(z) \qquad h(z_0) \neq 0, \ h \text{ analytic}$$

$$f(z) = w_0 + \frac{1}{(z-z_0)^N}\, g(z) \qquad g(z) = \frac{1}{h(z)}$$

$N \geq 1$ z_0 is a pole.

$N = 0$ z_0 is remov sing.

Math 4200
Wednesday November 4
3.3 Laurent series: classification of isolated singularities (Monday's notes) and multiplying Laurent series (today's notes).

Announcements: Quiz today!
Midterm next Friday will cover thru section 4.2 (The residue theorem).
Next homework assignment (due next Wednesday) is in today's notes.

Homework questions?

multiplying Laurent series term by term is legal:

We already know that we get the coefficient of $(z-z_0)^n$ in the Taylor series of a product $f(z)g(z)$ of analytic functions, by collecting the finite number of terms in the product of the Taylor series for f and g at z_0 which have that total power. The analogous statement is true for Laurent series, except that you may be collecting infinitely many terms. (You have a homework problem like this for Wednesday, 3.3.6.)

Theorem Let $f(z)$, $g(z)$ have Laurent series in $A := \{z \mid R_1 < |z-z_0| < R_2\}$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} b_m (z-z_0)^{-m} := \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$g(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k .$$

Then $f(z)g(z)$ has Laurent series

$$f(z)g(z) = \sum_{n=-\infty}^{\infty} d_n (z-z_0)^n$$

where

$$\boxed{d_n = \lim_{N\to\infty} \sum_{j=-N}^{N} a_j\, c_{n-j}.}$$

proof: Recall from Monday that we can recover the Laurent coefficients for an analytic function with a contour integral. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then Laurent coefficients for f are given by

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$b_m = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta - z_0)^{m-1}\, d\zeta,$$

which, if we write the Laurent series by combining the two sums as above,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

is the formula

$$a_n = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta - z_0)^{-n-1}\, d\zeta, \qquad n \in \mathbb{Z}$$

Thus, fixing n, the n^{th} Laurent coefficient d_n for $f(z)g(z)$ is given by

$$d_n = \frac{1}{2\pi i}\int_\gamma f(\zeta)g(\zeta)\left(\zeta - z_0\right)^{-n-1}d\zeta, \qquad n \in \mathbb{Z}.$$

Consider the truncated Laurent series for f, g,

$$f_N(z) := \sum_{j=-N}^{N} a_j\left(z - z_0\right)^j \quad , \quad g_N(z) = \sum_{k=-N+n}^{N+n} c_k\left(z - z_0\right)^k$$

which converge uniformly to f, g on the contour γ as $N \to \infty$, so

$$d_n = \lim_{N\to\infty} \frac{1}{2\pi i}\int_\gamma f_N(\zeta)g_N(\zeta)\left(\zeta - z_0\right)^{-n-1}d\zeta,$$

$$= \lim_{N\to\infty} \sum_{j,\,k=-N}^{N} \int_\gamma a_j\left(\zeta - z_0\right)^j c_k\left(\zeta - z_0\right)^k \left(\zeta - z_0\right)^{-n-1}d\zeta$$

by multiplication and term-by-term integration of the finite-sum truncated Laurent series. And picking off the non-zero integrals yields

$$d_n = \lim_{N\to\infty} \sum_{j=-N}^{N} a_j\, c_{n-j}.$$

Example (relates to hw problem 3.3.6):

a) The function $f(z) = \dfrac{e^{\frac{1}{z}}}{1 - z}$ is analytic for $0 < |z| < 1$. Find its residue at $z_0 = 0$.

$e^{\frac{1}{z}}$ $\frac{1}{1-z}$

b) Let γ be the circle of radiius $\dfrac{1}{2}$ centered at the origin. Find

$$\int_{\gamma} f(z)\, dz.$$

Math 4200-001
Homework 11
4.1-4.2
Due Wednesday November 11 at 11:59 p.m.
Exam will cover thru 4.2

4.1 1de, 3, 5, 7ab, 9
4.2 2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 9, 13.

w11.1 (extra credit) Prove Prop 4.1.7, the determinant computation for the residue at an order k pole for $f(z) = \dfrac{g(z)}{h(z)}$ at z_0, where $g(z_0) \neq 0$. (Hint: it's Cramer's rule for a system of equations.)